**Renormalization Group**

Now we’ll do NLσM model.

**RG Analysis of NLσM in 2+ε dimensions**

So we have seen that transverse fluctuations dominate systems at low temperature and destroy ordering. This suggests that we could do a perturbative expansion around d = 2 (the lower critical dimension), rather than d = 4 (usually the upper critical dimension). And we recall that d = 2 is the lower critical dimension for the localization phenomenon. But we also saw that we’d not want to try to work out the behavior of the Ising model near d = 2 because all the (even powered) φn terms would become relevant. Turns out the NLσM surmounts avoids this problem. So we return to the NLσM, expanded for small π,



where



and where K = βJ, and j = βh of course. So this gives us a theory similar to the φ4 theory already studied. Now let’s specialize to j(x) = j, a constant. Then we can write:



and we want to examine the renormalization of the coupling constants K, j. Before we do, I’m going to rewrite the integration term in the action by doing a little IBP, as this will aid writing out the Feynman rules,



which is:



Now we’ll split the fields into short and long,



where,



Now going to Fourier space,



Then we split π(k) into πshort(k) + πlong(k). And we get:



So now we can say:



where, e.g.,



and,



The perms are not all the same though. Perms are same if the we can switch vector indices and momentum labels to make them equivalent (since we can do this without changing the integral/sum). I’m gonna write out all the permutations, and color code the ones that are the same (only identical colors within the same group are same), along with the transformations that make them so. First I’ll do the K-interaction term,



So now we can say, not much nicer,



Now I’m going to write it with the 1/s! prefactor for every πa(s) and πb(s) (separately) in the term, to prep for the Feynman rules,



Yikes. Now I’ll do the j-interaction term, and simplify like last time,



and we get:



Well the top two can be made identical if do k1 → k2, k2 → k1. The bottom two can be made identical via k3 → k4, k4 → k3. In the middle block, the first two can be made identical via k3 → k4, k4 → k3 mapping. These can be made identical to the third via k1 → k2, k3 → k4 and vice versa. Then we have:



Now going to put in terms of the Feynman factor thing,



So with this in hand, we can formally calculate Z. Just like before, we’d say:



[SK refers to the Kπ4 vertex in Seff, Sj refers to the jπ4 vertex, and Sδ refers to the π2 vertex coming from the δ function variable change]. The outside blue integral is just a constant and so can be discarded. So we have:



where of course,



Then once we calculate < >short, will take its ln and put it in the exponent, and then try to work out the new renormalized action in terms of πℓ d.o.f.

**Feynman Rules**

Okay so now we’ll write down the rules for perturbatively calculating those diagrams that contribute to the < >short thing above. The first guy is the bare SK(πs) vertex. And the other ones are the terms in Sint(K)(πℓ,πs). The bars mark the legs whose momenta get counted in the associated mathematical expression. And for such purposes, momenta are counted positive if they are going radially away from the vertex.

Icon

Description automatically generated with medium confidence

and the j-vertices…the first guy is the bare j-vertex.

Diagram, schematic, radar chart

Description automatically generated

and the Sδ vertex,

A picture containing icon

Description automatically generated

Of course momentum must be conserved on it, and so k2 = -k1 (momenta pointing radially outwards). And the GF,

Diagram

Description automatically generated with low confidence

The dotted lines represent πs, and the solid lines πℓ. The solid πℓ lines are basically external lines as far as πs is concerned. Momentum is conserved at each vertex. The usual symmetry factors apply, as discussed in the path integrals folder for instance. Then we sum (1/Ld)Σk or integrate ∫ddk/(2π)d over all momenta, including those on the φℓ’s, but noting that the k’s in a φs will range from Λ/b → Λ, while k’s in a φℓ will ange from 0 → Λ/b. Well might note that since we factored the lattice spacing, *a*, out of our x position variable in the action, the sum would really be (1/N)Σk since k is now unitless. And we also sum over all indices a, b, etc. The diagrammatic expansion, out to first order in SK, Sj, Sδ, and Sint would look something like this, I think:

A picture containing chart

Description automatically generated

I might have missed a few vacuum bubbles in the first line. In the second line we have all the diagrams which renormalize the K and j terms in the free part of the action. The first renormalizes the K term, and the last three renormalize the j term. I’ve read that we don’t need to consider disconnected diagrams – these will cancel out, ultimately, when exponentiate into S. Well let’s do some calculations.

**zeroth order calculation**

So let’s take the first term in the expansion,

Text, letter

Description automatically generated

Then we have:



So now we need to see how we must rescale our πℓ variables to put Z back into its original form vis a vis the momentum integration. So apropos S(πℓ), we have:



First we need to rescale the momentum:



and we’ll get:



And we factor out the 1/b in the δ’s, and redefine the dummy variable of integration inside D[πℓ(k)] from πℓ(k/b) → πℓ(k). There would be a Jacobian associated with this change of variables, but it would just depend on b, and be a multiplicative constant for Z, and an additive constant for F, and so wouldn’t matter in the end.



Now we need this form to match the old one, but with renormalized coupling constants. Both prefactors need to be the same. So if we say π → ζπ, then we need, looking at the two K-terms,



Same goes if we’re looking at the two j-terms,



So we have:



This will introduce another Jacobian factor in D[π] of roughly bdΛ but this doesn’t matter, again.



which is:



Converting back to position space, we have, dropping the (ℓ) superscript, now that we’re in our final form:



So now it’s back in the desired form, and with renormalized coupling constants. So our renormalized coefficient so far would be:



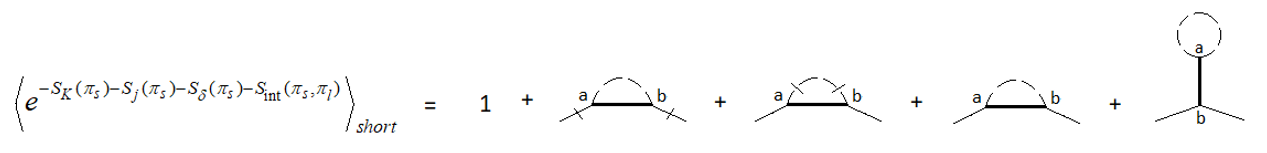
So at this order, we have λK = d-2, λj = d. So what would be our critical exponents at this point?



That doesn’t make much sense. So the Sint stuff clearly plays a large role.

**1st order calculation**

Let’s go out to out to first order. I’m going to ignore the vacuum bubbles as they won’t matter yet (or at all).



The first diagram is (no symmetry factor, as legs are distinct):

Diagram

Description automatically generated

which amounts to:



(b in integration bounds has no relation to b subscript in sum and π’s) And let’s see what will happen to this term when we rescale as we did in the previous section. So let’s do for the πℓ’s: k → k/b, πℓ(k/b) → πℓ(k), πℓ(k) → bdπℓ(k),



Now let’s convert to position space, and we can take the superscript off.



Next one,

Diagram

Description automatically generated

And we have,



(b in integration bounds has no relation to b subscript in sum and π’s) And let’s see what will happen to this term when we rescale as we did in the previous section. So let’s do for the πℓ’s: k → k/b, πℓ(k/b) → πℓ(k), πℓ(k) → bdπℓ(k),



Now let’s convert to position space, and we can take the superscript off.



Next one is,

Diagram

Description automatically generated

which gives us,



(b in integration bounds has no relation to b subscript in sum and π’s) And let’s see what will happen to this term when we rescale as we did in the previous section. So let’s do for the πℓ’s: k → k/b, πℓ(k/b) → πℓ(k), πℓ(k) → bdπℓ(k),



Now let’s convert to position space, and we can take the superscript off.



Almost there,

Diagram

Description automatically generated with medium confidence

So,



(b in integration bounds has no relation to b subscript in sum and π’s, and there’s a ½ symmetry factor for coincident propagator, and remember that there are only n-1 π’s) And let’s see what will happen to this term when we rescale as we did in the previous section. So let’s do for the πℓ’s: k → k/b, πℓ(k/b) → πℓ(k), πℓ(k) → bdπℓ(k),



Now let’s convert to position space, and we can take the superscript off.



So our total Z1 is:



Blue term can be written as:



So then Z1 simplifies to:



Now let’s fill this into our Z,



So the effective action so far is, dropping the ℓ sub/superscript.



Now as it turns out, in the large b limit, the two red terms cancel. This is because we’ll recall ∫0Λddq/(2π)d is just the continuum approximation of (1/N)Σq. But the sum over all q’s in the BZ should just be N, the number of unit cells in the lattice. And so we have ∫0Λddq/(2π)d = 1. I guess it’s okay that the bare π2 is renormalized away? So now we have:



Okay, well now we need to revisit our field scaling factor ζ. We’ve been using ζ = bd. And while this worked to zeroth order, it doesn’t now at first order. This is because our criterion for determining the field renormalization factor ζ was that the two K terms in S renormalized to the same thing. And right now they aren’t the same. But we can’t really compare the two because we have only looked at the renormalization of the Sint(K) term to zeroth order, while we’ve gone out to first order for the Sfree(K) term. There is a way around this. We can go back to the beginning in a previous file when we were writing our action for the NLσM. There were took our spins to be preferentially aligned with the field so that fluctuations, **π**, were perpendicular to it. But we could drop that presumption. All that would change is we’d have an extra term -∫ddx **j**(x)·**π**(x) = -∫ddk/(2π)d**j**(k)·**π**(k) in our action. As far as I can tell, there are no diagrams above that would contribute to the renormalization of this term to first order. So if we do as we’ve done for the previous πℓ’s: k → k/b, πℓ(k/b) → πℓ(k), πℓ(k) → bdπℓ(k), then this would come to, putting back in position space, -bd∫ddx **j**(x)·**π**(x). And our action, presently, would be:



So we can compare the two j-terms to first order. Under renormalization, they should have same prefactor since the magnet is isotropic. So scaling the fields π → ζπ, we have:



And we should have:



So then our renormalized K and j coefficients would be:



which is,



Well let’s turn these into differential equations. So we’ll let b = 1+ε. Anticipating that the ∫’s are already first order in ε, we can simplify a little, keeping terms only to O(ε).



and,



Okay let’s turn the first into an ODE.



Now let’s do the j term.



And all together, and specializing to dimensions slightly greater than 2 so defining d = 2+ε.



I’m going to switch notation L → b. Basically L will be the reference length, and b the scaling variable. But really I’m just changing notation,



Now we’ll look for fixed points. So setting the derivatives equal to zero,



Well these two equations seem to be incompatible. But we can obviate this issue by just saying j\* = 0. So let’s do that, and then plug this into the top equation,



So we have as our fixed point (and critical point, in this case):



So we see we can have a phase transition for dimensions above d = 2. This coincides with our knowledge that d = 2 is the lower critical dimension. Now we’ll linearize our equations around the fixed points,



and,



So altogether, and in terms of ΔK = K – K\*, and Δj = j – j\* = j,



We can solve these coupled equations. Could solve the bottom and then plug into the top. But I’ll use the general method outlined in the RG file. So we’ll write:



where ΔK = K – K\*, and Δj = j – j\* = j. As worked out in the RG file appendix, the solution is:



where λ and |λ> are the eigenvalues/eigenvectors of , which are:



So in our case, we have:



And so the solution is:



And separately, just out to O(ε)



where Δj1 = j of course. So what are λK and λj? The latter is clear. But the former isn’t really. For any finite Δj1, it seems Kb would scale predomantly with the last term, and so it’s exponent would be λK = λj. But I’ve read people say that we have to evaluate the exponent by examining the scaling when we set all other relevant variables to zero (don’t know *whyyyyy*). And if we set Δj1 = 0, then we’d clearly get λK = ε. I’ll just go with it, since that is the value we’re supposed to have.



and so various critical exponents are (to highest non-zero order):



and,



Getting some negative values for α in the small ε limit. I think a more careful analysis would show that we are now getting the next leading order contribution to α, and the leading order contribution would be a step function. Maybe the theory can’t capture step functions too well? Could argue this is just α = 0 of course. Whatever. And our actual exponents are (well, these are for the Ising model),

Table

Description automatically generated

Well whatever. It turns out that extrapolating to d = 3 gives very poor results, as compared to perturbing around the upper critical dimension. This problem persists for the localization phenomenon. We can plot the fixed point, eigenvectors, and typical flows – I guess I’m not doing the flows.

A picture containing text, antenna

Description automatically generated

**Random Comment**

Recall that in the O(n) model the longitudinal propagator was massfull resulting in exponentially decaying correlations in the longitudinal direction. But the transverse propagator was massless setting up long range correlations in that variable (which why it is these spin waves which are most important for critical phenomena).

**Another Random Comment**

If you review the work above, looks like we could just use the field renormalization ζ as ascertained at the level of the free field theory, i.e., ζ = bd, for purposes of determining whether more terms in the expansion of the small π action will scale to zero or not. Although it was later ascertained that we needed, altogether:



we can see that in the large b limit, this goes to const.×bd. So in the large b limit, the scaling, as established for the free theory, seems to hold.